Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 2, pp. 167–171, 2004

INTEGRALS OF MOTION OF AN INCOMPRESSIBLE FLUID OCCUPYING THE ENTIRE SPACE

V. V. Pukhnachev

UDC 532.5

This paper studies integral relations to which the solutions of the Navier–Stokes equations or Euler equations satisfy in the case of fluids filling the entire three-dimensional space. The existence of these relations is due to a rapid decrease of the velocity field at infinity (but not too rapid in order that the required asymptotic forms are reproduced with time). Of special interest are the integrals of motion whose density depends quadratically on the velocities or their derivative with respect to the coordinates. Such integrals (conservation laws) for the Navier–Stokes equations were recently found by Dobrokhotov and Shafarevich. In the present paper, new conservation laws are obtained, which are quadratic in the derivatives of the velocity and lead to identities that link the averaged and pulsation characteristics of free turbulent flows.

Key words: *Navier–Stokes equations, Euler equation, conservation laws.*

We assume that a viscous incompressible fluid occupies the entire space $\mathbb{R}^3$ and external volume forces do not act on it. The fluid velocity $\boldsymbol{v}(\boldsymbol{x}, t) = (v_1, v_2, v_3)$ and the fluid pressure $p(\boldsymbol{x}, t)$ are linked by the Navier–Stokes equations

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \Delta \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = 0, \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad 0 < t < T, \tag{1}$$

where $\nu \geqslant 0$ is the kinematic viscosity. The fluid density is set equal to unity without loss of generality. The fluid begins to move from a specified initial state:

$$\boldsymbol{v} = \boldsymbol{v}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad t = 0. \tag{2}$$

It is assumed that the function $\boldsymbol{v}_0$ satisfies the continuity equation $\nabla \cdot \boldsymbol{v}_0 = 0$ and some conditions of smoothness and decrease at infinity, which will be given below. For $\nu = 0$, system (1) becomes the Euler equations

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p, \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{3}$$

which describe the motion of an ideal incompressible fluid.

The resolvability of the Cauchy problem (1), (2) has been the subject of extensive research (see [1–3] and references therein) and is not considered here. We only recall that for the two-dimensional analog of the problem (1), (2), the existence and uniqueness theorem holds globally, i.e., for all $T > 0$ irrespective of the magnitude of the norm of the function $\boldsymbol{v}_0$ in the appropriate Banach space. For the three-dimensional problem, the global unique resolvability was proved under the additional assumption of axisymmetric motion [4], which implies that in cylindrical coordinates (r, θ, z), the velocity components v_r and v_z and the pressure p do not depend on θ and that $v_\theta = 0$. Generally, the existence of a single solution of problem (1), (2) in a small time interval $[0, T]$ was proved for arbitrary initial data from a certain class (for example, $\boldsymbol{v}_0 \in L^2(\mathbb{R}^3)$ for any $T > 0$ if the norm of the initial data is small.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 2, pp. 22–27, March–April, 2004. Original article submitted November 6, 2003.

Below, we assume that $\boldsymbol{v}_0 \to 0$ as $x \to \infty$. The order of decrease in $\boldsymbol{v}_0$ determines many qualitative properties of the solution of problem (1), (2), in particular, the presence or absence of time-independent functionals of its solution. The simplest of them is the momentum integral

$$\int \boldsymbol{v}\, dx = \int \boldsymbol{v}_0\, dx, \qquad t \in [0,T], \tag{4}$$

which is valid for any classical solution of the problem (1), (2) that satisfies the condition

$$|\boldsymbol{v}(\boldsymbol{x},t)| \leqslant C(1+|\boldsymbol{x}|)^{-\gamma}, \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad t \in [0,T] \tag{5}$$

with certain constants $\gamma > 3$ and $C > 0$ [the absence of the limits of integration in (4) and in the next relations implies that the integral is taken over the entire space].

The law of conservation of momentum (4) is equally valid for the motion of viscous and ideal fluids. However, the condition $\gamma > 3$ is very stringent and is not satisfied even for a simple solution of system (3) such as Hill's spherical vortex [5]. In cylindrical coordinates, in which the fluid is at rest at infinity, this solution has the form

$$v_r = -\frac{3Ua^3 r(z+Ut)}{2[r^2+(z+Ut)^2]^{5/2}}, \qquad v_z = -\frac{Ua^3[r^2-2(z+Ut)^2]}{2[r^2+(z+Ut)^2]^{5/2}}$$

$$\text{for} \quad r^2+(z+Ut)^2 \geqslant a^2, \quad v_\theta = 0 \quad \text{everywhere}, \tag{6}$$

$$v_r = -3Ua^{-2} r(z+Ut)/2, \qquad v_z = Ua^{-2}[6r^2+3(z+Ut)^2-5a^2]/2$$

$$\text{for} \quad r^2+(z+Ut)^2 \leqslant a^2,$$

where $a > 0$ and U are constants. In this case, the exponent $\gamma = 3$ in (5) and the integral (4) converges only in the sense of the principal value rather than absolutely. Since

$$\lim \int\limits_{B_r} \boldsymbol{v}\, dx = \lim \int\limits_{B_r} \boldsymbol{v}_0\, dx \qquad \text{at} \quad R \to \infty \tag{7}$$

(B_r is a sphere $|\boldsymbol{x}| < R$), expression (4) can be regarded as the time-independent fluid momentum $\boldsymbol{P}$.

A remarkable feature of the Hill solution is the localization of the vorticity: $\boldsymbol{\omega} \equiv \operatorname{rot} \boldsymbol{v} = 0$ outside the sphere $r^2+(z+Ut)^2 = a^2$. In such situations, the limits of expressions (7) for $R \to \infty$ can be calculated by the formula [5]

$$\boldsymbol{P} = \frac{1}{2}\int \boldsymbol{x} \times \boldsymbol{\omega}\, dx. \tag{8}$$

The definition of the fluid momentum (8) can also be fruitful for studies of the motion of a viscous fluid occupying the entire space if the vorticity in this motion is assumed to decrease rapidly as $\boldsymbol{x} \to \infty$ [6]. This situation takes place in the problem of the diffusion of Hill's spherical vortex with vanishing viscosity [7]. The problem is formulated as follows: it is required to find a solution of system (1) with the initial condition (2), where the vector $\boldsymbol{v}_0$ is determined by formulas (6) for $t = 0$. In this case, the function $\boldsymbol{v}_0$ is continuous but the vortex $\boldsymbol{\omega}_0 = \operatorname{rot} \boldsymbol{v}_0$ has a first-order discontinuity on the sphere $r^2+z^2 = a^2$. The presence of viscosity leads to instantaneous smoothing of the discontinuity, so that the solution of the problem (1), (2) becomes infinitely differentiable for $t > 0$. If $\nu \to 0$, this process is described by boundary-layer type functions. An unsteady boundary layer is concentrated near the moving sphere $r^2+(z+Ut)^2 = a^2$, and its thickness has order $(\nu t)^{1/2}$. Batishchev and Srubshchik [7] constructed the asymptotic form of the solution of the examined problem for $\nu \to 0$, which is valid in any finite time interval. This was done using the algorithm proposed in [8] to construct the asymptotic form of the solution of the problem of a planar fluid flow with initially localized vorticity for $\nu \to 0$. In [8], this algorithm was used to solve the problem of the effect of small viscosity on fluid flow in which the initial constant vorticity is concentrated inside an ellipse. In this case, the solution of the Cauchy problem for the two-dimensional Euler equations (3) describes motion in which the vortex region remains an ellipse that rotates at constant angular velocity around its center (so-called Kirchhoff's elliptic vortex [5]). In [7, 8], an estimate is given of the closeness of the exact and approximate solutions of the problem of the diffusion of Hill's spherical vortex and Kirchhoff's elliptic vortex at $\nu \to 0$ in any finite time interval $[0,T]$. We note that in both cases, the vorticity decreases exponentially when $\boldsymbol{x} \to \infty$ and $t \in [0,T]$. This guarantees that the momentum integral (8) and its planar analog converge and do not vary in time. If the vector $\boldsymbol{v}$ satisfies inequality (5) with $\gamma > 4$, the total moment of momentum of the fluid $\boldsymbol{M} = \int \boldsymbol{x} \times \boldsymbol{v}\, dx$ in the solution of the problem (1), (2) is conserved. However, the indicated condition is too stringent: assuming that it is satisfied for $t > 0$, we cannot, generally speaking, guarantee the reproducibility of the asymptotic form (5) with $\gamma > 4$ for $t > 0$. The exponent $\gamma = 4$ is critical [3]. If inequality (5) is satisfied for $\gamma = 4$ and if the initial vorticity is finite or decreases rapidly as $\boldsymbol{x} \to \infty$, then in the solution of the problem (1), (2) the quantity $\boldsymbol{M} = \dfrac{1}{2}\displaystyle\int \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{w})\, dx$, having the meaning of the resultant moment of momentum [5], is conserved.

The integrals of motion (4), (8) are linear in the velocities or their derivatives. It is of interest to find quadratic functionals conserved on the solutions of the Cauchy problem (1), (2). For an ideal fluid, such a functional is the energy integral

$$\int |\boldsymbol{v}|^2\, dx = \int |\boldsymbol{v}_0|^2\, dx, \qquad t \in [0, T] \tag{9}$$

[it is assumed that both integrals in (9) converge]. In a viscous fluid, the kinetic energy dissipates, and equality (9) becomes an inequality, which is strict if $\boldsymbol{v}_0 \neq 0$ for any $t \in (0, T]$. It is unexpected that for a rather wide class of motions of a viscous fluid, each of the kinetic energy components $\displaystyle\int v_k^2\, dx$ $(k = 1, 2, 3)$ decreases with time at the same rate. In [9], it was established that provided that $\boldsymbol{v}$ decreases rapidly as $\boldsymbol{x} \to \infty$, the following identities hold:

$$\int v_i^2\, dx = \int v_k^2\, dx, \qquad \int v_i v_k\, dx = 0 \qquad (i, k = 1, 2, 3; \quad i \neq k). \tag{10}$$

From the results of [9], it also follows that if in the solution of problem (1), (2) the functions $\boldsymbol{v}$, $\nabla \boldsymbol{v}$, and $\boldsymbol{v}_t$ decrease faster than $|\boldsymbol{x}|^{-4}$ as $\boldsymbol{x} \to \infty$, identities (10) necessarily hold at the initial time. This result can be treated as an instantaneous loss of the localization property of the velocity field in the solution of the Cauchy problem (1), (2). In [10], a similar result was obtained without any assumptions on the smoothness of the solution of the problem considered.

We give a simple derivation of identities (10) for the case of planar motion. We designate $x_1 = x$, $x_2 = y$, $v_1 = u$, and $v_2 = v$ and define the stream function ψ by the relations $u = \psi_y$ and $v = -\psi_x$. Then, as noted by Aristov, the momentum equation in the projection onto the x axis becomes

$$\frac{\partial}{\partial x}\,(p + \psi_y^2) = \frac{\partial}{\partial y}\,(\nu \Delta \psi - \psi_t + \psi_x \psi_y),$$

whence follows the existence of the function Q such that $p + \psi_y^2 = Q_y$ and $\nu \Delta \psi - \psi_t + \psi_x \psi_y = Q_x$. From Eqs. (1) and the definition of Q, it follows that this function satisfies the Poisson equation $\Delta Q = 2\psi_y \Delta \psi$. Multiplying both sides of the last equation by the linear function L of the variables x and y, integrating the resulting equality over the circle $D_r = \{x, y\colon\ x^2 + y^2 < R^2\}$, and performing simple manipulations, we obtain

$$\int\limits_{S_R} \Big(L\,\frac{\partial Q}{\partial R} - Q\,\frac{\partial L}{\partial R}\Big) R\, d\varphi = \int\limits_{D_R} [L_y(\psi_x^2 - \psi_y^2) - 2L_x \psi_x \psi_y]\, dx\, dy$$

$$+ \int\limits_{S_R} L[(\psi_y^2 - \psi_x^2)\sin\varphi + \psi_x \psi_y \cos\varphi] R\, d\varphi,$$

where S_R is the circle $x^2 + y^2 = R^2$ and $\varphi = \arctan\,(y/x)$. Assuming that the functions Q, ∇Q, and $\nabla \psi$ decrease rapidly with increase in R, we can pass to the limit $R \to \infty$ in the equality obtained. The limiting relation has the form

$$\int [L_y(\psi_x^2 - \psi_y^2) - 2L_x \psi_x \psi_y]\, dx\, dy = 0,$$

where the integral is taken over the entire plane (x, y). Setting $L = x \pm y$, we finally we obtain

$$\int (u^2 - v^2)\, dx\, dy = \int uv\, dx\, dy = 0,$$

which is the analog of identities (10) in the planar case.

Identities (10) have the meaning of integrals of motion for a viscous fluid occupying the entire space. However, they have a conditional nature because the existence of solutions of problem (1), (2) that decrease rapidly as $\boldsymbol{x} \to \infty$ is not guaranteed even if the initial data have this property. In particular, the condition $\boldsymbol{v} = O(|\boldsymbol{x}|^{-\gamma})$ for $\boldsymbol{x} \to \infty$, generally speaking, is not satisfied for $t > 0$ if $\gamma > 4$. Nevertheless, Brandolese [10] distinguished a class of initial functions $\boldsymbol{v}_0$ with a small norm for which inequality (5) is satisfied for all $T > 0$ with the exponent $\gamma \in (4, 5]$. This class is defined by the following conditions: v_{0i} is an odd function of x_i and an even function of the remaining variables; $v_{01}(x_1, x_2, x_3) = v_{02}(x_3, x_1, x_2) = v_{03}(x_2, x_3, x_1)$. The higher degree of symmetry of the initial data, associated with groups of regular polyhedra in $\mathbb{R}^3$, makes it possible to increase the critical value of γ but in the most favorable case, γ does not exceed seven (Brandolese's hypothesis). At the same time, for large $\boldsymbol{x}$, the asymptotic form of a typical velocity field $\boldsymbol{v}_0$ with a finite or rapidly decreasing function $\boldsymbol{\omega}_0 = \operatorname{rot} \boldsymbol{v}_0$ decreases as $O(|\boldsymbol{x}|^{-3})$ and its principal term corresponds to a dipole located at the coordinate origin. In this case, there are no grounds to hope that the integral identities (10) are satisfied. For example, for the initial data corresponding to Hill's spherical vortex (6), the value of the integral $\int (v_r^2 - 2v_z^2)\, dx$ at small t has the asymptotic form $-\pi U^2 a^3 [1 + O(\sqrt{\nu t}/a)]$, which is in conflict with (10). However, for such motions, too, integrals exist but their densities depend quadratically on the first and higher derivatives of the velocity components with respect to the space variables.

Our reasoning is based on the well-known relation, which is valid for the smooth solutions of system (1):

$$\Delta p = 2\Omega : \Omega - D : D, \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad 0 \leqslant t \leqslant T, \tag{11}$$

where D and Ω are the symmetric and antisymmetric parts of the tensor $\nabla \boldsymbol{v}$. We assume that $\partial v_i/\partial x_k = O(|\boldsymbol{x}|^{-\alpha})$ for $\boldsymbol{x} \to \infty$ and $t \in [0, T]$ $(i, k = 1, 2, 3)$, where $\alpha \geqslant 2$. Then, the following representations hold:

$$p = b(t)/|\boldsymbol{x}| + O(|\boldsymbol{x}|^{-2}) \quad \text{at} \quad \boldsymbol{x} \to \infty, \quad t \in [0, T], \qquad \nabla p = b(t) \nabla (1/|\boldsymbol{x}|) + O(|\boldsymbol{x}|^{-3}), \tag{12}$$

where

$$b(t) = \frac{1}{4\pi} \int (D : D - 2\Omega : \Omega)\, dx.$$

Representations (12) follow from the well-known properties of Newtonian's potential with rapidly decreasing density. Let us introduce the function $q = p + \boldsymbol{x} \cdot \nabla p$. From (12)it follows that $q = O(|\boldsymbol{x}|^{-2})$ as $\boldsymbol{x} \to \infty$. We can also show that

$$\nabla q = O(|\boldsymbol{x}|^{-3}) \qquad \text{at} \quad \boldsymbol{x} \to \infty, \quad t \in [0, T]. \tag{13}$$

We now define the vector function $\boldsymbol{w}$ by the equality $\boldsymbol{w} = \boldsymbol{x} \cdot \nabla \boldsymbol{v}$. It is important that this vector $\boldsymbol{w}$ is solenoidal (whereas the vector $\nabla \boldsymbol{v} \cdot \boldsymbol{x}$, generally speaking, does not have this property). Direct calculations show that if a pair $\boldsymbol{v}$ and p is a solution of Eq. (1), the functions $\boldsymbol{w}$ and $s = \boldsymbol{x} \cdot \nabla p$ satisfy the equations

$$\boldsymbol{w}_t + \boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v} = -\nabla s + \nu \Delta \boldsymbol{w}, \qquad \nabla \cdot \boldsymbol{w} = 0. \tag{14}$$

We apply the divergence operation to the first of Eqs. (14) and subtract the resulting equality from (11). Taking into account the equality $q = p + s$, we obtain Poisson's equation for the function q:

$$\Delta q = 2\Omega : (\Omega + \Psi) - D : (D + E), \qquad \boldsymbol{x} \in \mathbb{R}^3, \tag{15}$$

which is valid for all $t \in [0, T]$. In (15), E and Ψ are the symmetric and antisymmetric parts of the tensor $\nabla \boldsymbol{w}$. We assume, in addition, that the components of the vector $\boldsymbol{w}$ satisfy the condition $\partial w_i/\partial x_k = O(|\boldsymbol{x}|^{-\alpha})$ with $\alpha \geqslant 2$ for $\boldsymbol{x} \to \infty$ and $t \in [0, T]$ $(i, k = 1, 2, 3)$. Integrating equality (15) over the circle $|\boldsymbol{x}| < R$ and passing to the limit at $R \to \infty$, by virtue of (13) we have

$$\int [D : (D + E) - 2\Omega : (\Omega + \Psi)]\, dx = 0. \tag{16}$$

Identity (16) is the required integral of motion.

We now consider the solenoidal vector function $\boldsymbol{z} = \boldsymbol{x} \cdot \nabla \boldsymbol{w}$ and the associated "pressure" $n = \boldsymbol{x} \cdot \nabla s$. These functions satisfy the system

$$\boldsymbol{z}_t + \boldsymbol{w}_t + \boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + 2\boldsymbol{w} \cdot \nabla \boldsymbol{w} + \boldsymbol{z} \cdot \nabla \boldsymbol{v} = -\nabla n + \nu \Delta \boldsymbol{z}, \qquad \nabla \cdot \boldsymbol{z} = 0.$$

Postulating the estimates $\partial z_i/\partial x_k = O(|\boldsymbol{x}|^{-\alpha})$ for $\boldsymbol{x} \to \infty$, where $\alpha \geqslant 2$, and using the method described above, we arrive at the identity

$$\int [D : (E + G) + E : E - 2\Omega : (\Omega + \Phi) - 2\Psi : \Psi]\, dx = 0, \tag{17}$$

where G and Φ are the symmetric and antisymmetric parts of the tensor $\nabla \boldsymbol{z}$. The integral relation (17) contains the third derivatives of the functions v_i with respect to the variables x_k $(i, k = 1, 2, 3)$. The procedure of obtaining identities that are similar to (16) and (17) but include the higher-order derivatives of v_i can be continued but this is beyond the scope of the present paper.

In conclusion, we note that relations (10), (16), and (17), which are exact consequences of Navier –Stokes equations and some hypotheses on the nature of the decrease in the velocity field at infinity can be used to analyze free turbulence on the basis of Reynolds's hypotheses. Denoting the averaged and pulsation components of the functions v_i by $\bar{v}_i$ and v_i', respectively, and applying the averaging procedure to relations (10), we obtain

$$\int (\bar{v}_i^2 + \overline{v_i'^2}\,)\,dx = \int (\bar{v}_k^2 + \overline{v_k'^2}\,)\,dx, \qquad \int (\bar{v}_i \bar{v}_k + \overline{v_i' v_k'}\,)\,dx = 0 \qquad (i, k = 1, 2, 3;\ i \neq k). \tag{18}$$

Equalities similar to (18) can be obtained from relations (16) and (17). The equalities can be used to test semiempirical theories of free turbulent flows. A classical example of such motion is a turbulent vortex ring (see [6]).

The author thanks L. Brandolese, T. Miyakawa, and V. V. Shelukhin for fruitful discussions.

REFERENCES

1. O. A. Ladyzhenskaya, *Mathematical Problems of the Dynamics of a Viscous Incompressible Fluid* [in Russian], Nauka, Moscow (1970).
2. M. E. Schonbek, "Asymptotic behavior of solutions to free-dimensional Navier –Stokes equations," *Indiana Univ. Math. J.*, **41**, No. 3, 809–823 (1992).
3. T. Miyakawa "On space-time decay properties of nonstationary incompressible Navier–Stokes flows in R^n," *Funkcialaj Ekvacioj.*, **43**, No. 3, 541–557 (2000).
4. V. I. Yudovich and M. R. Ukhovskii, "Axisymmetric flows of ideal and viscous fluids occupying the entire space," *Prikl. Mat. Mekh.*, **32**, 59–69 (1968).
5. J. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1973).
6. B. A. Lugovtsov, "Turbulent vortex rings," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 38, Inst. of Hydrodynamics, Sib, Div., Russian Acad. of Sci., Novosibirsk (1979), pp. 71–88.
7. V. A. Batishchev and L. S. Srubshchik, "Asymptotic form of weak discontinuities of a fluid with vanishing viscosity," *Dokl. Akad. Nauk SSSR*, **197**, No. 5, 1038–1041 (1971).
8. L. S. Srubshchik and V. I. Yudovich, "Asymptotic form of weak discontinuities of a fluid with vanishing viscosity," *Dokl. Akad. Nauk SSSR*, **199**, No. 3, 563–566 (1971).
9. S. Yu. Dobrokhotov and A. I. Shafarevich, "On the behavior of the velocity field of an incompressible fluid at infinity," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 4, 38–42 (1996).
10. L. Brandolese, "On the localization of symmetric and asymmetric solutions of the Navier–Stokes equations in R^n," *Comp. Rend. Acad. Sci. Paris, Sér.* 1, **332**, 125–130 (2001).